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Functional Analysis Methods in the Study of the Optimal Transfer

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Introduction

IN the present Note the motion along the Earth and moon orbit in a given time of a space vehicle (acted upon by a low thrust) starting at the collinear libration points is analyzed. Let the interval $[0, T]$ be the duration of the vehicle motion and consider a linear space of the functions continuous on $(0,T)$. This space E consisting of L_1 $(0, T)$ absolutely integrable functions of the matrix of the fundamental solution for the uncontrolled system in a Banach space. The dual space E^* is the space of functionals, linear and continuous on $(0, T)$, representing the control vector. In E^* a norm is defined. Since the operator defined for the optimal transfer is bounded, the norm of this operator is the lower bound of the numbers satisfying the inequality given by the boundedness condition. Use is made of final condition for the considered problem to determine the control vector and the commutation time of the control.

Transfer in Vacuum

For a space vehicle acted upon by a low-thrust propulsion force, the transfer from the liberation points to the Earth or moon orbits is equivalent to a problem of encounter in orbits.

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Considering a rotating system having its origin at the collinear libration points for the Earth-moon system the linearized system of equations of motion of the space vehicle can be written in the form given in Eqs. (1).

$$\dot{x}_1 = x_2 \tag{1a}$$

$$\dot{x}_2 = K_1 x_1 + 2\omega x_4 + u_1 \tag{1b}$$

$$\dot{x}_3 = x_4 \tag{1c}$$

$$\dot{x}_4 = -2\omega x_2 + K_2 x_3 + u_2 \tag{1d}$$

where x_1, x_3 represent the coordinates of the space vehicle and x_2, x_4 are the velocity components; (u_1, u_2) being the thrust acceleration component in the rotating system.

The solution of the system represented by Eqs. (1) is written in the form

$$x_i(t) = \sum_{j=1}^{4} R_{ij}(t)x_j(0) + \sum_{j=1}^{4} \int_0^t K_{ij}(t,\tau)u_j(\tau)\,\mathrm{d}t \tag{2}$$

where $R_{ij}(t)$ is the matrix of the normal fundamental solutions at $t = 0$ of the system [Eqs. (1)] at $u_j = 0$. It follows

$$R_{ij}(0) = \delta_{ij} \tag{3}$$

We also denoted

$$K_{ij}(t,\tau) = \sum_{m=1}^{4} R_{im}(t)R_{mj}^{-4}(\tau) \tag{4}$$

R_{mj}^{-4} being the elements of the inverse fundamental matrix.

Functional Analysis Methods in Optimal Control

Let $[0, T]$ be the time interval on which we want to analyze the optimal transfer from the manifold

$$S_I = \{x \mid x_i(0) = x_i^i = 0 \qquad i = 1, \ldots 4\} \tag{5}$$

to the manifold

$$S_F = \{x \mid x_2(T) = c_2;\ x_4(T) = c_4\} \tag{6}$$

Let us consider the set

$$E = \{g \mid g = \{g_i(t)\} = [K_{i1}(T,t),\ K_{i2}(T,t)], \qquad i = 1, \ldots 4; \quad g_i \epsilon L_1(0,T)\} \tag{7}$$

Since $g_i\, L_1(0,T)$ is a Banach space it follows that E is a Banach space. Let us define in E the norm

$$\left\| g \right\| = \sum_{j=1}^{2} \int_0^T \left| g_j(t) \right| \tag{8}$$

The dual space of the linear and continuous functionals given by

$$E^* = \{u \mid u = \{u_j(t)\}; \qquad j = 1,2; \qquad u_j \epsilon L_\infty(0,T)\} \tag{9}$$

is also a Banach space that can be normalized by introducing the norm

$$\| u \| \max_{1 \le j \le 2} \max_{0 \le t \le T} |u_j(t)| \tag{10}$$

If $g \epsilon E$ and $u \epsilon E^*$ we may write

$$(g,u) = \sum_{j=1}^{2} \int_0^T g_j(t)u_j(t)\,\mathrm{d}t \tag{11}$$

Let F be a subspace of E containing al the linear combinations of g_i of the form

$$F = \{\phi \mid \phi = \sum_{i=1}^{4} \lambda_i g_i, \quad \lambda_i \epsilon R\} \tag{12}$$

Consider the linear functional $u_0 \in F$ uniquely determined by equations

$$(g_i, u_0) = c_i \qquad (i = 1, \ldots 4) \tag{13}$$

We may write

$$M = \|u_0\| = \max_{\phi \epsilon F} \frac{(\phi, u_0)}{\|\phi\|} = \max_{\lambda} \frac{\sum_{i=1}^{4} \lambda_i (g_i, u_0)}{\left\|\sum_{i=1}^{4} \lambda_i g_i\right\|}$$

$$= \max_{\lambda} \frac{\sum_{i=1}^{4} \lambda_i c_i}{\left\|\sum_{i=1}^{4} \lambda_i g_i\right\|} \tag{14}$$

By the Hahn-Banach theorem it follows that there exists a functional $u \epsilon E^*$ such that its restriction to F coincides with u_0 and

$$\|u\| = \|u_0\| = M \tag{15}$$

Assume that $\lambda_i^* (i = 1,...,4)$ are the values taken by the multipliers such that relation (14) holds. Taking into account Eq. (5), the solution (2) of the commanded system (1) is given by

$$x_i(t) = \int_0^t [S_{i1}(t-\tau)u_{,}(\tau) + S_{i2}(t-\tau)u_2(\tau)] \, d\tau \tag{16}$$

where, to shorten notation, we denote

$$R_{i,2,k}(t-\tau) = S_{ik}(t-\tau), \qquad k = 1,2 \tag{17}$$

From Eqs. (14) and (6) it follows that

$$M\left\|\sum_{i=1}^{4} \lambda_i^* g_i\right\| = \sum_{i=1}^{4} \lambda_i^* \int_0^T \sum_{j=1}^{2} S_{ij}(T-\tau)u_j(\tau) \, d\tau$$

$$= \sum_{j=1}^{2} \int_0^T \sum_{i=1}^{4} \lambda_i^* S_{ij}(T-\tau)u_j(\tau) \, d\tau$$

$$\leq \sum_{j=1}^{2} \int_0^T \left|\sum_{i=1}^{4} \lambda_i^* S_{ij}(T-\tau)\right| |u_j(\tau)| \, d\tau \tag{18}$$

using Eqs. (8), (10), and (15) we obtain

$$M\left\|\sum_{i=1}^{4} \lambda_i^* g_i\right\| = \sum_{j=1}^{2} \int_0^T \sum_{i=1}^{4} \lambda_i^* S_{ij}(T-\tau)u_j(\tau) \, d\tau$$

$$\leq M \sum_{j=1}^{2} \int_0^T \left|\sum_{i=1}^{4} \lambda_i^* S_{ij}(T-\tau)\right| d\tau = M\left\|\sum_{i=1}^{4} \lambda_i^* g_i\right\| \tag{19}$$

which implies

$$\sum_{i=1}^{4} \lambda_i^* S_{ij}(T-\tau)u_j = M\left|\sum_{i=1}^{4} \lambda_i^* S_{ij}(T-\tau)\right| \tag{20}$$

Equation (20) leads to

$$v_j(\tau) = M \operatorname{sgn}\left[\sum_{i=1}^{4} \lambda_i^* S_{ij}(T-\tau)\right] \qquad (j = 1,2) \tag{21}$$

The optimal transfer such that $x_i \epsilon s_F$ is realized by determining a control vector $u \epsilon E^*$ with the minimum norm. The inspection of Eq. (14) shows that its fulfillment is equivalent to

$$M^{-1} = \min_{\lambda_i} \left\|\sum_{i=1}^{4} \lambda_i g_i\right\| = \min_{\lambda_i} \left\{\sum_{j=1}^{2} \int_0^T \left|\sum_{i=1}^{4} \lambda_i S_{ij}(T-\tau)\right| d\tau\right\} \tag{22}$$

such that

$$\sum_{i=1}^{4} \lambda_i^* c_i = 1 \tag{23}$$

Determination of Extremals

Since the final position of the space vehicle at the evolution time T is not yet determined and hence $x_1(T)$ and $x_3(T)$ are unknown, in order to satisfy Eq. (23) we choose $\lambda_1^* = \lambda_3^* = 0$, $\lambda_2^* = \lambda^*$, $\lambda_4^* = (1 - c_2\lambda^*/c_4)$ where λ^* is a parameter whose value will follow from the imposed conditions. Performing calculations in Eq. (21) we are led to

$$u_1(\tau) = M \operatorname{sgn}\left\{\lambda^* [(\alpha^2 - K_2) ch\, \alpha(T-\tau) + (b^2 + K_2) \cos b(\tau - \tau)] + \frac{1 - c_2\lambda^*}{c_4}(-2\omega)[\alpha sh\, \alpha(T-\tau) + b \sin b(T-\tau)]\right\} \tag{24a}$$

$$u_2(\tau) = M \operatorname{sgn}\left\{\lambda^* \left[\frac{(4\omega^2 - b^2 - K_2)(\alpha^2 - K_2)}{2\alpha} sh\, \alpha(T-\tau) + \frac{(4\omega^2 - K_2 + \alpha^2)(b^2 + K_2)}{2b} \sin b(T-\tau)\right] + \frac{1 - c_2\lambda^*}{c_4} 2\omega[-(4\omega^2 - b^2 - K_2) ch\, \alpha(T-\tau) + (4\omega^2 - K_2 + \alpha^2) \cos b(T-\tau)]\right\} \tag{24b}$$

It is noticed that if $0 < t_1 < T$ we may choose $\lambda_2 = \lambda^*$ and $\lambda_4^* = (1 - c_2\lambda^*/c_4)$ such that $u_1(t_1) = 0$. It follows $\lambda^* > 0$.

Taking $1 - c_2\lambda^* > 0$, from the anlysis of expressions (24) it is remarked that there exists $\lambda^* \epsilon [(0, 1/c_2)]$ for which one may obtain the values of the components of the control vector on the considered time interval

$$u_1 = \begin{cases} -M, t \epsilon [0, t_1) \\ +M, t \epsilon (t_1, T] \end{cases} \tag{25a}$$

$$u_2 = +M, t \epsilon [0, T] \tag{25b}$$

The transfer from the collinear libration points correspond to the transfer from the manifold S_I to the manifold S_F, which comes to satisfy conditions

$$M\left[-\int_0^{t_1} R_{2k,2}(T-\tau) \, d\tau + \int_{t_1}^{T} R_{2k,2}(T-\tau) \, d\tau + \int_0^T R_{2k,4}(T-\tau) \, d\tau\right] = c_{2k} \; (k = 1,2) \tag{26}$$

From Eq. (26) we determine the commutation time t_1 of the command and the modulus of the command vector M. Taking into account the values of the command components given by Eqs. (25), solution (16) may be written as

$$x_i(t) = M\left[-\int_0^{t_1} R_{i2}(t-\tau) \, dt + \int_{t_1}^{t} R_{i2}(t-\tau) \, d\tau + \int_{t_1}^{t} R_{i4}(t-\tau) \, d\tau\right] \; (i = 1, \ldots 4) \tag{27}$$

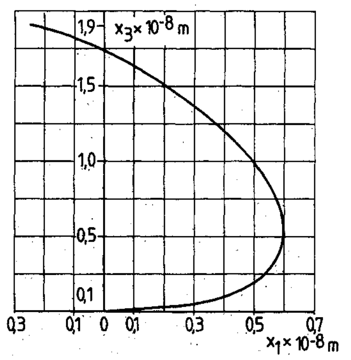

Fig. 1 Optimal transfer trajectory.

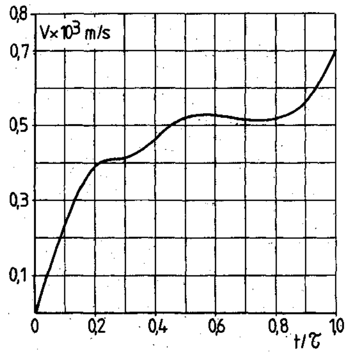

Fig. 2 Variation of the velocity.

where $x_1(t)$ and $x_3(t)$ represent the parametric form of the trajectory for the considered problem.

Numerical Application

On the basis of the analytical solution [Eq. (27)] a numerical application was performed concerning the optimal transfer from the libration point L_2 of the Earth-moon system.

It was taken into account that, for the Earth-moon system, $m^* = 0.01227$. The initial data used are $x_j(0) = 0$ ($j = 1, \ldots, 4$).

We mention that the Eqs. (27) were calculated in units $D = 1$ where $D = 3.84 \times 10^8$ m represents the Earth-moon distance and the time unit τ is chosen such that the gravitational parameter of the Earth is 1. The obtained results have been subsequently transformed in units m/s (Figs. 1 and 2). The evolution time was taken $T = \tau = 3.77 \times 10^5$ s and the components of the final velocity $c_2 = 400$ m/s and $c_4 = 600$ m/s.

Conclusions

This functional analysis study performed for the formulated transfer problem revealed the existence of piecewise constant controls. Their calculation indicates values $M = 2.25 \times 10^{-4}$ g, which means that for fuel durations $T \geq 10^5$ s consistent with the hypothesis of small thrust, the amount of the acceleration due to the thrust is situated between the admissible limits of 10^{-3}–10^{-6} g. The obtained commutation time of the command is $t_1 = 2.13 \times 10^5$ s.

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Bang-Bang Control of Flexible Spacecraft Slewing Maneuvers: Guaranteed Terminal Pointing Accuracy

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I. Introduction

FUTURE spacecraft may be quite flexible compared with their predecessors. Many proposed applications require slewing these vehicles between two quiescent attitudes. Bang-bang control of rest-to-rest slewing maneuvers has been the focus of a number of recent investigations.[1-7] In these works the time- and fuel-optimal[1] and the time-optimal[3-7] problems are considered. Fixed-time maneuvers minimizing a measure of spillover energy have also been considered.[2] In some cases it is possible to obtain closed-form expressions for post-time-optimal control spillover measures (spillover energy and maximum postmaneuver pointing error, for example).[5] The purpose of this Note is to extend these results to any bang-bang, rest-to-rest slewing maneuver for a class of flexible spacecraft. The main contribution of this work is that the bounds obtained here allow us to determine, a priori (i.e., before any sequence of switching times is computed), the number of flexible modes to be controlled actively during a bang-bang maneuver in order to guarantee a prespecified pointing accuracy for an infinite dimensional evaluation model.

II. Equations of Motion

We will consider an unconstrained, cylindrical rigid central body to which $N (N \geq 2)$ identical flexible appendages are rigidly and symmetrically attached. The spacecraft is to be controlled by a single torque actuator located at the rigid central body. The appendage displacements and slopes are assumed small relative to the undeformed appendage; appendages are assumed inextensible, and only planar motions are considered. The appendage deformations relative to their undeformed shapes are assumed identical and antisymmetric; no structural damping is assumed. Lastly, it is assumed that the rigid central body rotation rate remains small at all times. Henceforth, we shall use uppercase boldface type to denote matrices and lowercase boldface type to indicate vectors. The following coupled ordinary differential equations are obtained after discretization (using the assumed mode method, for example) and truncation[4,5]:

$$J_0\ddot{\theta} + \boldsymbol{m}^T\ddot{\boldsymbol{q}} = T, \qquad \boldsymbol{m},\boldsymbol{q} \in \mathbb{R}^M \tag{1}$$

$$N\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} + \boldsymbol{m}\ddot{\theta} = 0, \qquad \boldsymbol{m},\boldsymbol{q} \in \mathbb{R}^M \tag{2}$$

where $\theta(t)$ is the rigid-body angular position, $\boldsymbol{q}(t)$ is the appendage generalized coordinate vector, $T(t)$ is the rigid-body control torque, J_0 is the total undeformed rotational inertia of the vehicle, $\boldsymbol{K}$ is a diagonal matrix whose ith diagonal entry is $N\tilde{\omega}_i^2$ ($\tilde{\omega}_i$ is the frequency of the ith bending mode), and $\boldsymbol{m}$ is defined according to

$$\{\boldsymbol{m}\}_i \triangleq N\int_0^L \rho(x)(R + x)\phi_i(x)\,\mathrm{d}x, \qquad i = 1,2,\ldots, M \tag{3}$$

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